On the periodicity of Somos sequences

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Somos–(4) sequence $\{s_n\}$ is defined by initial data

$$s_1 = s_2 = s_3 = s_4 = 1$$

and recurrence relation

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2.$$

It begins with

$$\dots, 2, 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \dots$$

(Obviously $s_n = s_{5-n}$.)

Somos–(4)

1
2
3
7
23
59
314
1529
8209
83313
620297
7869898
126742987
1687054711
47301104551
1123424582771
32606721084786
1662315215971057
61958046554226593
4257998884448335457
334806306946199122193
23385756731869683322514
3416372868727801226636179
397068399296019083272746599
62595441409640805744780870839
15886280085653574502219650145963
2993270782370572835241882188306602
1107464108502549897934954766675333353
424202695779047673359251734568172798797
157131202095317153373302215985417166354641
118253616021740752904870890844242082115408921
67590197098545828288675045139752541981367236186
58001627211098227368565907864525353503139727123291
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Somos–(6)

Somos first introduced the sequence Somos-(6) such that

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 1$$

and

$$s_{n+3}s_{n-3} = s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

He raised the question whether all the terms are integer:

1470*. *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \geq 6$. Computer calculations show that $a_6, a_7, \ldots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

Somos M. Problem 1470. Crux Mathematicorum, 15: 7 (1989), p. 208.

Integrality

The integrality of Somos–(4) and Somos–(5) was proved by Janice Malouf, Enrico Bombieri and Dean Hickerson (1990).

The integrality of Somos–(6) was proved by Dean Hickerson (April 1990).

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Somos–(8) is a sequence with initial data

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = 1$$

satisfying recurrence relation

$$s_{n+4}s_{n-4} = s_{n+3}s_{n-3} + s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

Somos–(8) is NOT an integer sequence:

$$\ldots, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, \frac{420514}{7}, \ldots$$

Somos–(8) is a wild object. Probably it has no properties at all.

Somos sequences

Definition

For integer $k \geq 4$ Somos–k sequence is a sequence generated by quadratic recurrence relation of the form

$$s_{n+k}s_n = \sum_{j=1}^{[k/2]} \alpha_j s_{n+k-j}s_{n+j},$$

where α_j are constants and $s_0, \ldots, s_{k-1}$ are initial data.

In particular Somos-4 is defined by initial data s_0, s_1, s_2, s_3 and fourth-order recurrence

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2,$$

Integrality condition

Theorem (Fomin and Zelevinsky, 2002)

For a Somos–k sequences ($k = 4, 5, 6$ and 7) all of the terms in the sequences are Laurent polynomials in these initial data whose coefficients are in $\mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}]$, so that

$$s_n \in \mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}, s_1^{\pm 1}, \dots, s_k^{\pm 1}] \text{ for all } n \in \mathbb{Z}.$$

Fomin S. and Zelevinsky A. “The Laurent Phenomenon”, Adv. Appl. Math. 28 (2002) 119–144.

Integrality of original Somos–(k) sequences follows from the theorem with

$$\alpha_1 = \dots = \alpha_{[k/2]} = s_1 = \dots = s_k = 1.$$

But this Theorem is not a final step.

A sigma-function solution for Somos–4

Elementary proofs don't spread any light on the nature of Somos sequences. There are elliptic curves hidden behind Somos–4–5 and hyperelliptic curves of genus 2 behind Somos–6–7.

The general solution of Somos–4 recurrence relation is given by C. Swart (2003) and A. Hone (2005)

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where $z, z_0 \in \mathbb{C}^*$, and

$$\sigma(z) = \sigma_\Gamma(z) = z \prod_{w \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2}$$

is Weierstrass sigma-function associated to plane lattice Γ.

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Finite rank sequences

The main property of Somos–4 and Somos–6: they have finite rank.

Definition (1)

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank r if maximal rank of two infinite matices

$$M_0 = (s_{m+n}s_{m-n})\Big|_{m,n=-\infty}^{\infty}, \qquad M_1 = (s_{m+n+1}s_{m-n})\Big|_{m,n=-\infty}^{\infty}$$

is r.

Definition (2)

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank r if r is a least possible k such that for all integer m and n

$$s_{m+n}s_{m-n} = \sum_{j=1}^{k} f_j(m)g_j(n), \qquad s_{m+n+1}s_{m-n} = \sum_{j=1}^{k} \widetilde{f}_j(m)\widetilde{g}_j(n).$$

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is r.

Definition (3)

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank if (s_n) is a Somos-k sequence for every $k \geq k_0$.

Finite rank sequences: examples

Example (1)

Let $s_n = AB^n C^{n^2}$. Then

$$s_{m+n}s_{m-n} = A^2 B^{2m} C^{2m^2+2n^2} = f(m)g(n),$$

where

$$f(m) = A^2 B^{2m} C^{2m^2} \quad \text{and} \quad g(n) = C^{2n^2}.$$

(And almost the same for $s_{m+n+1}s_{m-n}$.) So the sequence $s_n = AB^n C^{n^2}$ has rank 1. And this is the only rank-1 sequence.

Example (2)

The sequence $s_n = n$ has rank 2 because

$$s_{m+n}s_{m-n} = m^2 - n^2, \quad \text{and} \quad s_{m+n+1}s_{m-n} = m(m+1) - n(n+1).$$

Finite rank sequences: examples

Example (3)

$s_n = n^k$ has rank $k+1$.

Example (4)

$s_n = \sin n$ has rank 2: $\sin(m+n)\sin(m-n) = \frac{1}{2}\left(\cos 2m - \cos 2n\right).$

Example (5)

For Fibonacci sequence $s_n = F_n$ we have

$$F_{m+n}F_{m-n} = \frac{1}{5}\left(L_{2m} - (-1)^{m+n}L_{2n}\right),$$

$$F_{m+n+1}F_{m-n} = \frac{1}{5}\left(L_{2m+1} - (-1)^{m+n}L_{2n+1}\right),$$

where $L_n = F_{n-1} + F_{n+1}$ are Lucas numbers. So Fibonacci sequence has rank 2 as well.

Finite rank sequences: non-trivial examples

Example (6)

The sequence $s_n = \sigma(z_0 + zn)$ has rank 2 because of addition formula

$$\sigma(u-v)\sigma(u+v) = -\wp(u)\sigma(u)^2\sigma(v)^2 + \wp(v)\sigma(u)^2\sigma(v)^2.$$

Example (7)

The same for

$$s_n = \theta_j(z_0 + zn)$$

because we know addition theorems of the form

$$\theta_1(y+z)\theta_1(y-z)\theta_4^2 = \theta_3^2(y)\theta_2^2(z) - \theta_2^2(y)\theta_3^2(z)\dots$$

Finite rank sequences: non-trivial examples

Example (8)

The Riemann theta function

$$\theta(z,\tau) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}\langle\tau m,m\rangle+\langle m,z\rangle\right)},$$

where $z \in \mathbb{C}^g$ and the imaginary part of a $g \times g$ matrix τ is positive definite, has rank 2^g.

Finite rank sequences: Somos examples

Example (Ma, 2010; conjectured by Gosper & Schroeppel, 2007)

Somos-4 has rank 2.

It follows from general formula

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}}.$$

Example (Fedorov, Hone, 2016; conj. by Gosper & Schroeppel, 2007)

Somos-6 has rank 4.

It follows from general formula for s_n in terms of the Kleinian sigma-function of genus two.

Theorem (Fedorov, Hone, 2016)

The general solution of Somos–6 recurrence has the following form

$$s_n = AB^n C^{n^2} \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where σ is the Kleinian sigma-function associated with some genus 2 algebraic curve X, and $z, z_0 \in \mathrm{Jac}(X)$.

Somos-4: the main property

Magic determinant

General Somos-4 has rank 2. It means that Somos–4 sequence satisfy “magic determinant property” or “addition formula” (Ma, 2010)

$$\begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} & \begin{vmatrix} s_{m_1+n_1}s_{m_1-n_1} & s_{m_1+n_2}s_{m_1-n_2} & s_{m_1+n_3}s_{m_1-n_3} \\ s_{m_2+n_1}s_{m_2-n_1} & s_{m_2+n_2}s_{m_2-n_2} & s_{m_2+n_3}s_{m_2-n_3} \\ s_{m_3+n_1}s_{m_3-n_1} & s_{m_3+n_2}s_{m_3-n_2} & s_{m_3+n_3}s_{m_3-n_3} \end{vmatrix} = 0, \end{array}$$

where m_i, n_i $(i = 1, 2, 3)$ are arbitrary integers or half-integers. It is the main property because another properties of Somos-4 sequence follow from “magic determinant”.

Doubling formulae for Somos-(4)

$$\begin{array}{c} \\ n \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} n & 1 & 0 \end{array} \\ \begin{vmatrix} s_{2n}s_0 & s_{n+1}s_{n-1} & s_n^2 \\ s_{1+n}s_{1-n} & s_2 s_0 & s_1^2 \\ s_n s_{-n} & s_1 s_{-1} & s_0^2 \end{vmatrix} \end{array} = 0,$$

$$\begin{vmatrix} s_{2n+1}s_0 & s_{n+2}s_{n-1} & s_{n+1}s_n \\ s_{2+n}s_{1-n} & s_3 s_0 & s_2 s_1 \\ s_{n+1}s_{-n} & s_2 s_{-1} & s_1 s_0 \end{vmatrix} = 0,$$

— imply integrality;
— allow to calculate $s_{mn} \pmod p$ from $(s_{n-3}, s_{n-2}, s_{n-1}, s_n)$ in $O(\log(m))$ operations.

Diffie-Hellman key exchange with Somos-4 sequences
[Gosper, Orman, Schroeppel]

- Alice chooses A from $[1; p-1]$
- Alice calculates $(s_{A-3}, s_{A-2}, s_{A-1}, s_A) \pmod p$
- Bob does the same with B
- Alice sends $(s_{A-3}, s_{A-2}, s_{A-1}, s_A) \pmod p$ to Bob
- Bob sends $(s_{B-3}, s_{B-2}, s_{B-1}, s_B) \pmod p$ to Alice
- Alice computes $s_{BA} \pmod p$
- Bob computes $s_{AB} \pmod p$

General properties of finite rank sequences

Properties which seemed to be more or less equivalent for general Somos sequences (different sides of Integrality):

- Finite rank
- Laurent phenomenon
- Positivity
- Reasonable tropicalization
- Periodicity $\pmod{N}$
- General formula in terms of theta-functions

Periodicity of Somos sequences

Somos–(4) sequence $\{s_n\}$ is defined by initial data

$$s_1 = s_2 = s_3 = s_4 = 1$$

and recurrence relation

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2.$$

Periodicity of Somos–(4) sequence is not obvious because we can not find all elements from the recurrence relation

$$s_{n+2}s_{n-2} \equiv s_{n+1}s_{n-1} + s_n^2 \pmod{m}.$$

Sometimes we have to calculate $0/0 \pmod{m}$.

Let $E = \{(x, y) : y^2 = x^3 + ax + b\}$ be an elliptic curve and $P = (x, y) \in E$. Then the point

$$P_n = nP = \underbrace{P + P + \ldots + P}_{n}$$

has a very special coordinates. There is a $\psi_n = \psi_n(x, y) \in \mathbb{Z}[a, b, x, y]$ such that $nP = \left(\frac{\cdots}{\psi_n^2}, \frac{\cdots}{\psi_n^3}\right)$. First elements are

$$\begin{gathered}\psi_0 = 0, \quad \psi_1 = 1, \quad \psi_2 = 2y, \quad \psi_3 = 3x^4 + 6ax^2 + 12bx - a^2, \\ \psi_4 = 4y(x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - 8b^2 - a^3).\end{gathered} \tag{1}$$

Polynomials $\psi_n(x, y)$ are known as *division polynomials*. They satisfy the following recurrence

$$\psi_1^2\psi_{n+2}\psi_{n-2} = \psi_2^2\psi_{n+1}\psi_{n-1} - \psi_1\psi_3\psi_n^2 \qquad (n \geq 3). \tag{2}$$

For any particular curve E and point $P \in E$ we get a sequence of numbers.

Ward (1948) proved that elliptic divisibility sequences modulo an arbitrary prime p are periodic and estimated the period length.

Ayad (1993), under certain technical restrictions, proved the periodicity of an elliptic divisibility sequence modulo an arbitrary number.

Shipsey (2000) proved that elliptic divisibility sequences are periodic modulo modulo p^2.

Silverman (2005) established the periodicity in finite fields.

Theorem (Robinson, 1992)

If general Somos-k sequence satisfy Laurent property, if $s_0, \dots, s_{k-1}$ are prime to m, and if

$$s_{r+n} \equiv s_n \pmod{m} \quad \textit{for} \quad n = 0, 1, \dots, k-1,$$

then $\{s_n\}$ has a period r.

The proof follows directly from Laurent property:

$$s_n = \frac{p_n(s_0, \dots, s_{k-1})}{q_n(s_0, \dots, s_{k-1})},$$

where $q_n(s_0, \dots, s_{k-1})$ is a monomial and it is prime to m.
From this theorem follows that Somos-(4) and Somos-(5) are periodic.
But his method does not work for $k = 6$ and $k = 7$.

The identity

$$\begin{array}{c} \\ n \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{matrix} \quad k & \qquad 1 & \qquad 0 \end{matrix} \\ \begin{vmatrix} s_{n+k}s_{n-k} & s_{n+1}s_{n-1} & s_n^2 \\ s_{1+k}s_{1-k} & s_2 s_0 & s_1^2 \\ s_k s_{-k} & s_1 s_{-1} & s_0^2 \end{vmatrix} = 0, \end{array}$$

means that Somos-4 is Somos-k for arbitrary even $k \geq 4$: for some α_k, β_k

$$s_{n+k}s_{n-k} = \alpha_k s_{n+1}s_{n-1} - \beta_k s_n^2.$$

The identity

$$\begin{array}{c} \\ n \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} k & 1 & 0 \end{array} \\ \begin{vmatrix} s_{n+k+1}s_{n-k} & s_{n+2}s_{n-1} & s_n^2 \\ s_{2+k}s_{1-k} & s_3s_0 & s_2s_1 \\ s_{k+1}s_{-k} & s_2s_{-1} & s_1s_0 \end{vmatrix} \end{array} = 0,$$

means that Somos-4 is Somos-k for arbitrary odd $k \geq 5$: for some $\tilde{\alpha}_k$, $\tilde{\beta}_k$

$$s_{n+k+1}s_{n-k} = \tilde{\alpha}_k s_{n+2}s_{n-1} - \tilde{\beta}_k s_{n+1}s_n.$$

In particular

$$s_{n+2}s_{n-3} = -s_{n+1}s_{n-2} + 5s_ns_{n-1}.$$

The integrality of Somos-(4)

For Somos-(4) we have

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2,$$
$$s_{n+2}s_{n-3} = -s_{n+1}s_{n-2} + 5s_ns_{n-1},$$

so

$$s_{n+2} = \frac{s_{n+1}s_{n-1} + s_n^2}{s_{n-2}},$$
$$s_{n+2} = \frac{-s_{n+1}s_{n-2} + 5s_ns_{n-1}}{s_{n-3}},$$

and $(s_{n-2}, s_{n-3}) = 1$. So if $s_0, s_1, \dots, s_{n+1} \in \mathbb{Z}$, then $s_{n+2} \in \mathbb{Z}$.

The periodicity of Somos-(4)

Two recurrences

$$s_{n+2}s_{n-2} \equiv s_{n+1}s_{n-1} + s_n^2 \pmod{p^\gamma},$$
$$s_{n+2}s_{n-3} \equiv -s_{n+1}s_{n-2} + 5s_ns_{n-1} \pmod{p^\gamma}$$

allow to find $s_{n+2} \pmod{p^\gamma}$ once we know 5 previous elements: aither $(s_{n-2}, p) = 1$ or $(s_{n-3}, p) = 1$.

The periodicity of Somos-(6)

For Somos-(6) we can find three recurrences

$$s_{n+5}s_{n-4} = -s_{n+4}s_{n-3} - s_{n+3}s_{n-2} + s_{n+2}s_{n-1} + 34s_{n+1}s_n,$$
$$s_{n+5}s_{n-5} = s_{n+4}s_{n-4} + 15s_{n+2}s_{n-2} - 19s_{n+1}s_{n-1} + 34s_n^2,$$
$$s_{n+5}s_{n-6} = s_{n+3}s_{n-4} + 19s_{n+2}s_{n-3} + 34s_{n+1}s_{n-2} + 19s_ns_{n-1}.$$

Moreover, for any prime p one of the elements s_{n-4} or s_{n-5} or s_{n-6} must be coprime to p. Why? Because

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 1$$

and

$$s_{n+3}s_{n-3} = s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

Theorem (AU, 2023)

Let $\{s_n\}$ be a finite rank integer Somos sequence. Then for any positive integer m the sequence $\{s_n \mod m\}$ is eventually periodic.

Reminder:

Definition (3)

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank if (s_n) is a Somos-k sequence for every $k \geq k_0$.

First idea: we can generate arbitrarily many recurrence relations.

Second idea:

Lemma

Let $\{s_n\}$ be a finite rank integer Somos sequence, and let p be a prime. Then one of the following two conditions is satisfied:

(a) For any $m \geq 1$, there exists a $T_m \geq 0$ such that $s_n \equiv 0 \pmod{p^m}$ for $|n| \geq T_m$.

(b) There exist $m \geq 0$, $T \geq 0$, and $N > 0$ such that $s_n \equiv 0 \pmod{p^m}$ for $|n| \geq T$ but at least one of any N consecutive elements of the sequence (s_n) is not divisible by p^{m+1}.

A natural candidate for further investigations is the Gale–Robinson sequence generated by

$$s_{m+n}s_m = \alpha s_{m+r}s_{m+n-r} + \beta s_{m+p}s_{m+n-p} + \gamma s_{m+q}s_{m+n-q},$$

where $r + p + q = n$.