Medial axes and the reaching of singularities

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Selected Topics in Mathematics

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1. The medial axis

Consider $\varnothing \neq X = \overline{X} \subsetneq \mathbb{R}^n$ and let $d(z, X)$ denote the Euclidean distance from $z \in \mathbb{R}^n$ to X. Put

$$m(z) = m_X(z) := \{x \in X \mid ||z - x|| = d(z, X)\}.$$

Definition (Blum, 1967)

We call $m\colon \mathbb{R}^n \to \mathcal{P}(X)$ the *multifunction of closest points*. Its multivaluedness set

$$M_X := \{z \in \mathbb{R}^n \mid \#m(z) > 1\} \subset \mathbb{R}^n \setminus X$$

is the *medial axis* (formally we should be adding: 'of $\mathbb{R}^n \setminus X$').

Remark

By the strict convexity of the norm, $\operatorname{int} M_X = \varnothing$.
If X is **definable** (e.g. *semi-algebraic*) or *subanalytic*, then so is M_X (Denkowski 2011) in which case we also have $\operatorname{int}\overline{M_X} = \varnothing$.
Outside tame geometry this may not be true (Fremlin 1997).

The medial axis

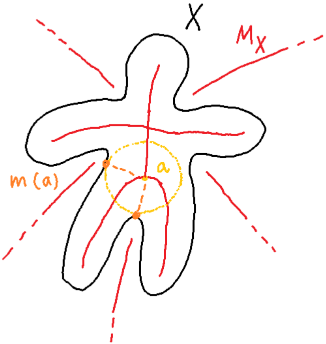


Figure: The medial axis M_X of a subset $X \subset \mathbb{R}^2$ and the set of closest points $m(a)$ of a point $a \in M_X$ with the sphere $\mathbb{S}(a, d(a, X))$ in yellow.

2. Preliminaries

Motzkin Theorem (1935)

$M_X \neq \varnothing \ \Leftrightarrow \ X$ is non-convex.

Some notation:
We write $\mathbb{B}(x,r) = \mathbb{B}_n(x,r) \subset \mathbb{R}^n$ for the open Euclidean ball
$\mathbb{S}(x,r) := \partial\mathbb{B}(x,r)$ for the sphere.
The *Peano tangent cone* of X at $a \in X$ is the 3rd Whitney cone:

$$C_a(X) = \{v \in \mathbb{R}^n \mid \exists X \ni x_\nu \to a, t_\nu > 0\colon\ t_\nu(x_\nu - a) \to v\},$$

and the *Clarke normal cone* of X at a,

$$N_a(X) = \{w \in \mathbb{R}^n \mid \forall v \in C_a(X), \langle v, w\rangle \leq 0\}.$$

We put $V_a(X) := N_a(X) \cap \mathbb{S}^{n-1}$ for *the unit normal directions*.

Remark

$a \in m(z) \ \Rightarrow \ z - a \in N_a(X).$

Cones

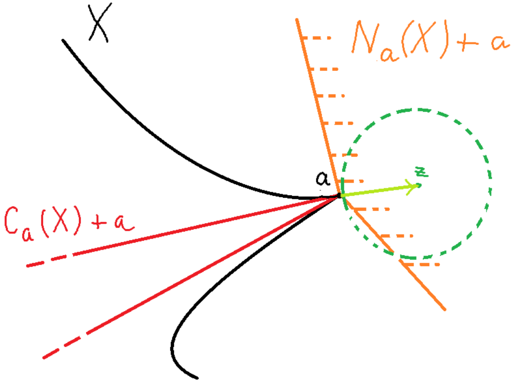


Figure: The tangent cone $C_a(X)$, the normal cone $N_a(X)$ and the vector $z - a \in N_a(X)$.

Regular and singular points

Recall the different classes of *regular and singular points*:

$$\mathrm{Reg}_k X := \{x \in X \mid (X,x) \text{ is } \mathcal{C}^k-\mathrm{smooth}\},\ \mathrm{Sng}_k X := X \setminus \mathrm{Reg}_k X,$$

for $k \in \mathbb{N} \cup \{\infty, \omega\}$ where $\mathcal{C}^\omega$ denotes analycity (then we write $\mathrm{Reg} X := \mathrm{Reg}_\omega X$ and $\mathrm{Sng} X := X \setminus \mathrm{Reg} X$ for the singularities.)

Example

For a plane analytic curve $\Gamma \subset \mathbb{R}^2$ through the origin we have $0 \in \mathrm{Sng}\Gamma$ if and only if either Γ has a cusp at zero, or there is an integer $k \geq 1$ such that $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$. All the possibilities can occur:
Take two relatively prime integers $p > q$ with q odd and such that for a given k, we have $k < p/q < k+1$ and consider the curve Γ defined by $y^q = x^p$. Then $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$. For instance the function $y = x^{5/3}$ has analytic graph and is $\mathcal{C}^1$ but not $\mathcal{C}^2$ smooth at the origin.

The squared distance function

Poly-Raby Theorem (1984)

Let $X \subset \mathbb{R}^n$ be a closed, nonempty set and $\delta(z) := \mathrm{dist}(z, X)^2$. Then for any $k \geq 2$ or $k \in \{\omega, \infty\}$,

$$\mathrm{Reg}_k X = \{z \in \mathbb{R}^n \mid \delta \textit{ is of class } \mathcal{C}^k \textit{ in a neighbourhood of } z\} \cap X.$$

We have to assume here $k \geq 2$ as is easily seen from the example of $X = (-\infty, 0]$ in $\mathbb{R}$.

Theorem (Clarke 1975, Yomdin 1981, Birbrair-Denkowski 2017)

M_X coincides with the set of non-differentiability points of $\delta(z)$.

3. The Nash Lemma and its generalizations

Nash Lemma (1952)

Let X be a $\mathcal{C}^k$-submanifold of an open set $\Omega \subset \mathbb{R}^n$ where $k \geq 2$, or $k \in \{\infty, \omega\}$. Then there exists an arbitrarily small neighbourhood $U \subset \Omega$ of X such that

- **(i)** $m|_U$ is univalued i.e. each point $x \in U$ has a unique closest point $m(x) \in X$;
- **(ii)** the function $m\colon U \ni x \mapsto m(x) \in X$ is of class $\mathcal{C}^{k-1}$, or, respectively, $\mathcal{C}^k$ with $k \in \{\infty, \omega\}$.

The general singular counterpart of the Nash Lemma is the following theorem. Given $X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ we denote by X_t its section at the point t i.e. the set $\{x \in \mathbb{R}^n \mid (t, x) \in X\}$. Let $\pi_k(t, x) = t$.

Definable version with parameters

Theorem (Denkowski 2011)

Let $\varnothing \neq X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be definable with locally closed t-sections and $Y := \pi_k(X)$. Then there exists a definable set $W \subset \mathbb{R}^k \times \mathbb{R}^n$ with open t-sections and such that $X_t \subset W_t$ is closed in W_t and $m_t(x) \neq \varnothing$ for $x \in W_t$, where
$m_t(x) := \{y \in X_t \colon ||x - y|| = \mathrm{dist}(x, X_t)\}, \quad (t, x) \in W$, and

1. the multifunction $m(t, x) := m_t(x)$ is definable;
2. If $M_t = \{x \in W_t \mid \#m(t, x) > 1\}$, then the set $M := \bigcup_{t \in Y}\{t\} \times M_t \subset W$ is definable with nowheredense sections M_t and in particular $m \colon W \setminus M \to \mathbb{R}^n$ is a definable function;
3. for any integer $p \geq 2$, there is a definable set $F^p \subset W$ containing M and such that each F^p_t is closed and nowheredense; moreover, $X_t \setminus F^p_t = \mathrm{Reg}_p X_t$ and $m(t, \cdot)$ is $\mathcal{C}^{p-1}$ in a neighbourhood of $x \in W_t \setminus \overline{M_t} \Leftrightarrow\ x \notin F^p_t$.

• The Rolin-Le Gal result on the existence of o-minimal structures that do not admit $\mathcal{C}^\infty$ cellular cell decompositions implies that we cannot expect to take $p = \infty$ in the theorem above.

• When the parameter t is fixed we recognize here the multifunction $x \mapsto m(t, x)$ of the closest points to the set X_t. The section M_t of M is the set of non-unicity (multivaluedness) of this multifunction. In (3) this set is extended to a set 'eating away' the singularities of class $\mathcal{C}^p$ of the set X_t; this extension is defined by the class $\mathcal{C}^{p-1}$ of the function $m(t, \cdot)$.

• Everything here depends in a definable way on the multidimensional parameter t. This is no longer true in the subanalytic case:
$X = \{(x, 1/x) \mid x > 0\} \cup \bigcup_{n=1}^{+\infty}\{(1/n, -n)\} \subset \mathbb{R} \times \mathbb{R}$ is subanalytic, but the set $M = \bigcup\{(1/n, 0)\}$ is not.

Subanalytic version

Theorem (Denkowski 2011)

Let $X \subset \mathbb{R}^n$ be subanalytic, nonempty and locally closed. Then there exists a subanalytic neighbourhood $W \supset X$ in which X is closed and

1. the multifunction $m(x) = \{y \in X \colon ||x - y|| = \mathrm{dist}(x, X)\} \neq \varnothing$, for $x \in W$, is subanalytic;
2. the set $M_X = \{x \in W \colon \# m(x) > 1\}$ is subanalytic and nowheredense (in particular $m \colon W \setminus M_X \to \mathbb{R}^n$ is a globally subanalytic function);
3. there is a nowheredense, subanalytic set $F \subset W$ closed in W and such that $M_X \subset F$, $F \cap X = \mathrm{Sng} X$ and $x \in W \setminus \overline{M_X}$ is a point of analyticity of m if and only if $x \in W \setminus F$.

4. The reaching of singularities problem

The Nash Lemma implies that

$$\overline{M_X} \cap X \subset \mathrm{Sng}_2 X.$$

For any point $x \in X$ such that $x \in \overline{M_X}$, the medial axis is said to *reach the singularity* x. Clearly,

$$\mathrm{Sng}_2 X = \mathrm{Sng_1X} \cup (\mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X)$$

the union being disjoint.
We put

$$X^* := \mathrm{Sng_1X}, \quad X^\circ := (\mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X).$$

Problem

Characterise $\overline{M_X} \cap X = (\overline{M_X} \cap X^*) \sqcup (\overline{M_X} \cap X^\circ)$.

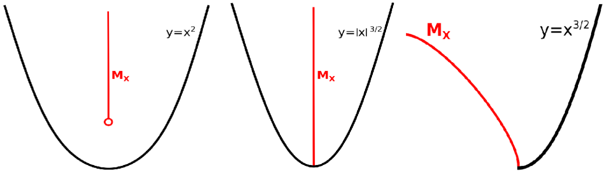
Some examples

Consider $\mathbb{R}^2$ and the following sets:

- $X\colon y=|x|$ yields $X^*\cap\overline{M_X}=\{(0,0)\}=X\cap\overline{M_X}$;
- $X\colon y\geqslant|x|$ yields $X^*\cap\overline{M_X}=\varnothing=X\cap\overline{M_X}$;
- $X\colon y=|x|^{3/2}$ yields $X^\circ\cap\overline{M_X}=\{(0,0)\}=X\cap\overline{M_X}$;
- $X\colon y=(1+\mathrm{sgn} x)x^2$ yields $X^\circ=\{(0,0)\}$ but $(0,0)\notin\overline{M_X}$.

The approaching of M_X brings along also an interesting additional metric information about the singularity it reaches.

A most surprising example is that of $X\colon y^2=x^3$ and even more that of a single branch $X\colon y=x^{3/2}, x\geq 0$.



The Tangent Cone Criterion for X^*

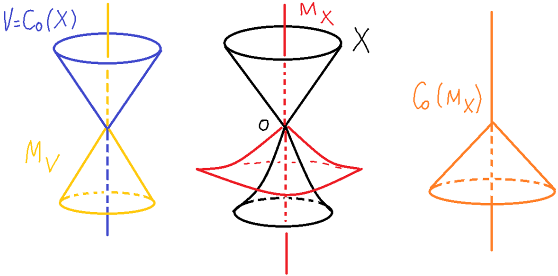
The following criterion works in the definable case:

Theorem (Birbrair-Denkowski 2017)

Assume $C_0(X)$ is non-convex. Then $0 \in X^* \cap \overline{M_X}$ and, moreover,

$$C_0(\overline{M_X}) \supset \overline{M_{C_0(X)}}.$$

We seldom have more than an inclusion, see the fancy vodka-glass example: $X\colon \{z = \sqrt{x^2 + y^2}\} \cup \{x^2 + y^2 + z^3 = 0\}$.



General tool: the reaching radius

Definition (Birbrair-Denkowski 2017)

We define the *weak reaching radius*

$$r'(a) = \inf_{v \in V_a(X)} r_v(a)$$

where $r_v(a) = \sup\{t \geq 0 \mid a \in m(a + tv)\}$ is the *directional reaching radius* (or *v-reaching radius*). Next we put

$$\tilde{r}(a) = \liminf_{X \setminus \{a\} \ni x \to a} r'(x)$$

for the *limiting reaching radius*. Finally, we define the *reaching radius* as

$$r(a) = \begin{cases} r'(a), & a \in \mathrm{Reg}_2 X, \\ \min\{r'(a), \tilde{r}(a)\}, & a \in \mathrm{Sng}_2 X. \end{cases}$$

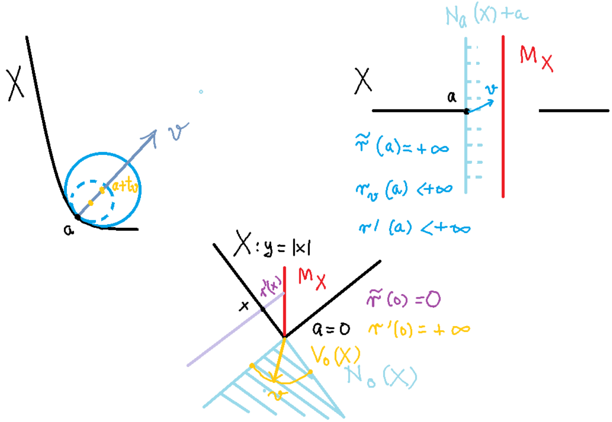


Figure: *Left:* inflating a ball in the direction v; *right:* infinite limiting radius, but finite directional radius; *bottom:* finite limiting radius, but infinite directional radius.

Vanishing of the reaching radius

The definition does not require any definability assumption.

Theorem (Birbrair-Denkowski 2017)

Without any definability assumptions on X we have
$X \cap \overline{M_X} = r^{-1}(0)$.
Moreover, if X is definable, so is the function $r\colon X \to [0, +\infty]$.

A major role in the proof is played by the so called *proximal inequality*. We say that $v \in V_a(X)$ is proximal for X, if for some $r > 0$, $m(a + rv) = \{a\}$. This is equivalent to the following inequality:

$$(\#) \qquad \exists r > 0\colon \forall x \in X,\ \langle x - a, v\rangle \leq \frac{1}{2r}||x - a||^2.$$

Further properties of the reaching radius

Theorem (Białożyt 2021)

1. $r(a) = \liminf_{X \ni x \to a} r'(x)$ (allowing stationary sequences);
2. r is lower semi-continuous so that $X \cap \overline{M_X}$ is closed;
3. On $\mathrm{Reg}_2 X$ the function $r = r'$ is continous;
4. If $a \in \mathrm{Reg}_2 X$, then $V_a(X) \ni v \mapsto r_v(a)$ is continous.

Even though the definition seems complicated, the reaching radius is computable and thus applicable.
This is a generalization of the radius of curvature to singular sets.
It is related to Federer's reach. A similar but weaker notion was also investigated by Miura for $\mathcal{C}^1$ smooth hypersurfaces in 2016.

Several criteria of A. Białożyt

Theorem

1. (Inner semi-continuity Criterion, Białożyt 2021)
$C_a(X) \not\subset \liminf_{X\ni x\to a} C_x(X) \ \Rightarrow a \in X^* \cap \overline{M_X}$;
2. (Topological criterion, Białożyt 2021) If there is a topological manifold $\Gamma \subset X$ such that $\dim \Gamma = \dim_a X$ and $a \in \Gamma \setminus \overline{M_X}$, then $(\Gamma, a) = (X, a)$ and (X, a) is a $\mathcal{C}^{1,1}$ submanifold;
3. (LNE criterion, Białożyt 2024) If $(X, 0)$ is non-LNE, then $0 \in X^* \cap \overline{M_X}$.

Note that $0 \in X^\circ \ \Rightarrow \ (X, 0)$ is LNE.
Here LNE stands for *Lipschitz normally embedded* meaning the inner metric d_X =*infimum of lengths of rectifiable arcs joining two points in* X and the Euclidean one restricted to X are equivalent.
(3) is based on the fact that $m(x)$ is locally Lipschitz on $\mathbb{R}^n \setminus \overline{M_X}$.

Superquadratic points

The notion of *superquadratic points* introduced by Birbrair and Denkowski for $\mathcal{C}^1$-smooth hypersurfaces is motivated by the situation met with in $\mathbb{R}^2$. It was carried over to $\mathcal{C}^1$-smooth points of any dimension by Białożyt thanks to the fact that a submanifold is locally a graph over its tangent space.

Definition

X is called *superquadratic at* $a \in \mathrm{Reg}_1 X$ if the *shape function*

$$g_X(\varepsilon) := \max\{||x - \pi(x)|| \mid x \in X\colon ||\pi(x) - a|| = \varepsilon\},\ 0 \leqslant \varepsilon \ll 1,$$

where $\pi\colon \mathbb{R}^n \to (T_a X + a)$ is the orthogonal projection onto the tangent space $T_a X$ translated to a, satisfies

$$\frac{g_X(\varepsilon)}{\varepsilon^2} \to +\infty\ (\varepsilon \to 0^+).$$

In a polynomially bounded o-minimal structure, such a function g_X is definable and can be written as $g_X(\varepsilon) = c\varepsilon^\eta + o(\varepsilon^\eta)$ with $c, \eta > 0$, provided its germ at the origin is non-zero which is the case, if $a \in \mathrm{Sng}_2 X$. If $a \in X^\circ$, we have $\eta > 1$. Superquadraticity means precisely that $\eta < 2$.
Since the exponent η is just the order of vanishing $\mathrm{ord}_0 g_X$ of g_X at the origin, it can be easily shown that in fact, assuming for simplicity $a = 0$ and writing $\mathbb{R}^n = T_0X \oplus (T_0X)^\perp$ with the corresponding coordinates $x = u \oplus v$,

$$\mathrm{ord}_0 g_X = \max\{\theta > 0 \mid (X, 0) \subset \{u \oplus v \mid ||v|| \leqslant \mathrm{const.} ||u||^\theta\}\}.$$

Put $\mathcal{SQ}(X) := \{x \in \mathrm{Reg}_1 X \mid X \text{ is superquadratic at } x\}$. Then $\mathcal{SQ}(X) \subset X^\circ$ and the following theorem holds (Birbrair-Denkowski, 2017, for $\mathrm{codim} X = 1$, Białożyt, 2021, for the general case).

Theorem

$$\mathcal{SQ}(X) \subset X^\circ \cap \overline{M_X}.$$

Moreover, if $n = 2$ or $\dim X = 1$, then the converse inclusion holds as well.

In general, there is no equality.

Example (Birbrair-Denkowski)

Consider $X = \{z = y|x|^{3/2}\}$ which is the graph of a $\mathcal{C}^1$ function $z = f(x, y)$ in $\mathbb{R}^3$. We easily check that $\mathrm{ord}_0 f \geq 2$ so that X is not superquadratic at the origin, but as it is such along all the other points of the y-axis, we have $0 \in \overline{M_X}$ by the Theorem above.

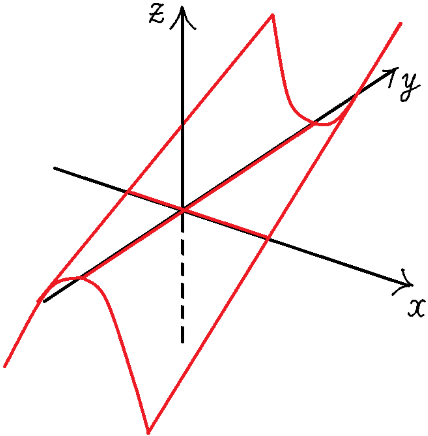


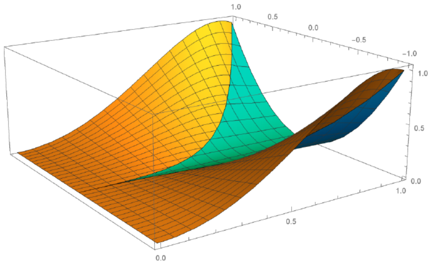
Figure: The surface $X = \{z = y|x|^{3/2}\}$, here $0 \in \overline{\mathcal{SQ}(X)} \setminus \mathcal{SQ}(X)$.

Example (Białożyt)

Let X be the graph of the $\mathcal{C}^1$ function

$$f(x,y) = \begin{cases} \frac{y^2}{x}, & |y| < x^3, x > 0; \\ 2x^2|y| - x^5, & |y| \geq x^3, x > 0; \\ 0, & x \leq 0. \end{cases}$$

It is not superquadratic at any point and yet $0 \in \overline{M_X} \cap X$ as X contains a suitable part of a rotated cone.



5. Plane case

In the plane case the problem is completely solved.

Preparatory Lemma

If $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected, i.e. X has a single branch ending at the origin, then the tangent cone $C_0(X)$ is a half-line that we can identify with $\mathbb{R}_+ \times \{0\}^{n-1} \subset \mathbb{R}^n$ in properly chosen coordinates and X is near zero the graph of a definable $\mathcal{C}^1$ function $f\colon [0, \varepsilon) \to \mathbb{R}$ with $f(0) = 0$ and $f'(0) = 0$.

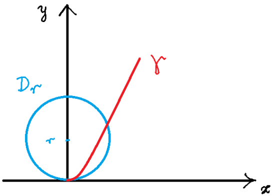
X as in the Lemma above is superquadratic at zero iff $f \not\equiv 0$ and $\mathrm{ord}_0 f < 2$.
The definability of f allows us also to assume that f has constant convexity on $[0, \varepsilon)$ and is $\mathcal{C}^2$ on $(0, \varepsilon)$.

The following Lemma accounts for the name *superquadratic*:

Lemma on superquadraticity

If $\gamma\colon [0,\varepsilon) \to [0,+\infty)$ is superquadratic with $\gamma(0) = \gamma'(0) = 0$, then for any $r > 0$ the disc $D_r := \mathbb{B}((0,r),r) \subset \{y > 0\}$ tangent to the x-axis at zero contains points of γ inside.

Proof: It follows from the obvious observation that if $g\colon [0,r) \to \mathbb{R}_+$ denotes the usual parametrization of the lower part of the circle ∂D_r through zero, then $g(x) = \frac{1}{2r}x^2 + o(x^2)$ near zero. At the same time $\gamma(x) = ax^\alpha + o(x^\alpha)$ with $a > 0$ and $\alpha \in (0,2)$ and so there must be $g(x) < \gamma(x)$ for small x.



Proposition (Birbrair-Denkowski 2017

Assume that $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected. Then $0 \in \overline{M_X}$ if and only if X is superquadratic at zero.

Proof: If X is superquadratic at zero, then by Lemma on superquadraticity, the weak reaching radius $r'(0,0)$ is zero and so the reaching radius $r(0,0)$ is zero, too. By the REaching Radius Theorem, it means that $0 \in \overline{M_X}$.
If X is not superquadratic at zero, then either $f \equiv 0$, or $\alpha \geq 2$, where f is the function from the Preparatory Lemma. In both cases f has a $\mathcal{C}^2$ extension by 0 through zero and the Nash Lemma leads to the conclusion that $0 \notin \overline{M_X}$.

Dimension Lemma

If $X \subset \mathbb{R}^2$ is definable with $\dim_0 X = 1$ and $0 \in \overline{M_X} \cap X$, then $\dim_0 M_X = 1$.

Proof: Since the assumptions imply that $0 \in \overline{M_X} \setminus M_X$, then by the Curve Selection Lemma, $\dim_0 M_X \geq 1$. On the other hand, M_X has empty interior, whence $\dim_0 M_X < 2$.

Proposition (Birbrair-Denkowski 2017)

Assume that X is as in the previous Proposition and $0 \in \overline{M_X} \cap X$. Then the tangent cone $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero. To be more precise, if X near zero is the graph of $f\colon [0, \varepsilon) \to \mathbb{R}$ and f is, say, convex, then $C_0(M_X) = \{0\} \times [0, +\infty)$.

If we are dealing with a $\mathcal{C}^1$-smooth curve, a so called 'rolling disc' argument yields:

Theorem (Birbrair-Denkowski 2017)

Assume that $0 \in X^\circ$. Then $0 \in \overline{M_X}$ iff X is superquadratic at the origin.

In the presence of at least two branches, we have the following result for a $\mathcal{C}^1$-singularity:

Theorem (Birbrair-Denkowski 2017)

Let $X \subset \mathbb{R}^2$ be a definable curve with $0 \in X^*$ and assume that the germ $(X \setminus \{0\}, 0)$ has at least two connected components. Then $0 \in \overline{M_X}$.

Complete solution in $\mathbb{R}^2$

If $(X,0) \subset \mathbb{R}^2$ is a definable pure one-dimensional closed germ, then $X \setminus \{0\}$ consist of finitely many branches $\Gamma_0, \dots, \Gamma_{k-1}$ ending at zero and dividing a small ball $\mathbb{B}(0,r)$ into k regions. For $k > 1$, if we enumerate the branches in a consecutive way, we can call these open regions $D(\Gamma_i, \Gamma_{i+1})$, $i \in \mathbb{Z}_k$. Assuming that $0 \in \overline{M_X}$, we say that a pair of *consecutive* branches Γ_i, Γ_{i+1} *contributes* to M_X at zero, if $0 \in \overline{M_X \cap D(\Gamma_i, \Gamma_{i+1})}$.

Let $1 \leq c \leq k$ be the number of contributing regions. For each such region $D(\Gamma_i, \Gamma_{i+1})$ we have two half-lines ℓ_i, ℓ_{i+1} tangent to Γ_i, Γ_{i+1} at zero, respectively. These half-lines define an oriented angle $\alpha(i, i+1) \in [0, 2\pi]$, consistent with the region.

Note that it may happen that $\alpha(i, i+1) = 2\pi$; indeed, if X consists of the two branches $\Gamma_0 = [0, +\infty) \times \{0\}$ and the superquadratic $\Gamma_1 = \{y = x^{3/2}, x \geq 0\}$, then both regions $D(\Gamma_0, \Gamma_1)$ and $D(\Gamma_1, \Gamma_0)$ are contributing. The angles are 0 and 2π, respectively.

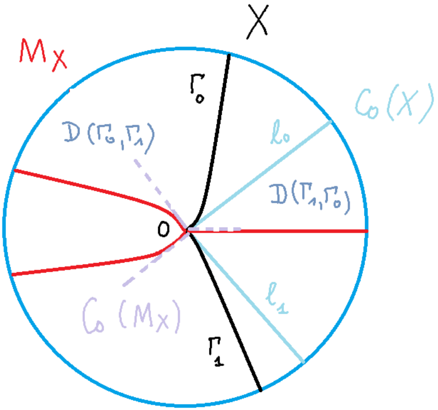


Figure: An example with two branches, two contributing regions and three branches for the medial axis.

As we know that M_X is one-dimensional, the germ $(\overline{M_X}, 0)$ consists of finitely many branches ending at zero. For a definable curve germ $(E, 0)$, we will denote by $b_0(E)$ the number of its branches at the origin.
Taking into account the previous results, we have the following solution:

Theorem (Birbrair-Denkowski 2017)

Assume that $0 \in \overline{M_X} \cap X$ where X is a pure one-dimensional closed definable set in the plane. Then,
(1) either $b_0(X) = 1$, in which case $b_0(M_X) = 1$ and $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero,

(2) or $b_0(X) = k > 1$, in which case $b_0(M_X) \leq c + 1$ where c is the number of contributing regions, and $C_0(M_X)$ is the union of the bisectors of all the pairs of half-lines forming up $C_0(X)$ given by pairs of consecutive branches delimiting regions that contribute to M_X at zero with possibly one exception:
there is at most one contributing region $D(\Gamma_i, \Gamma_{i+1})$ with angle $\alpha(i, i+1) > \pi$ in which case at least one of the curves Γ_i, Γ_{i+1} is superquadratic at zero and $M_{i,i+1} = M_X \cap D(\Gamma_i, \Gamma_{i+1})$ has at most two branches at zero and $C_0(M_{i,i+1})$ consists of one or two half-lines orthogonal to the corresponding tangent ℓ_i or ℓ_{i+1}.

The need for taking $c + 1$ in (2) is illustrated by the following example shown in the previous picture.

Example (see the last Figure)

Rotate the superquadratic curve $y = x^{3/2}$, $x \geq 0$ by $\pi/6$ anticlockwise and the curve $y = -x^{3/2}$, $x \geq 0$ by the same angle clockwise, obtaining two curves Γ_0, Γ_1 with tangent half-lines at zero $y = (1/\sqrt{3})x, x \geq 0$ and $y = -(1/\sqrt{3})x, x \geq 0$, respectively. Let $X = \Gamma_0 \cup \Gamma_1$. Then we have two contributing regions: $D(\Gamma_1, \Gamma_0)$ with $\alpha(1,0) = \pi/3$ and $D(\Gamma_0, \Gamma_1)$ with $\alpha(1,2) = 5\pi/3$. The medial axes has three branches at zero: the half-line $[0, +\infty) \times \{0\}$ and two curves symmetric with respect to $(-\infty, 0] \times \{0\}$, living in the quadrants $\{x \leq 0, y \geq 0\}$ and $\{x \leq 0, y \geq 0\}$, respectively. Then

$$C_0(M_X) = ([0, +\infty) \times \{0\}) \cup \{y = -\sqrt{3}x, x \leq 0\} \cup \{y = \sqrt{3}x, x \leq 0\}.$$

6. New approach: directional and general curvature

We assume the closed, definable germ $(X,0) \subset \mathbb{R}^n$ satisfies the following assumption: the affine hull of the Peano tangent cone of X at zero is of the form

$$(*) \qquad \mathrm{Aff}(C_0(X)) = \mathbb{R}^k \times \{0\}^{n-k},$$

where $1 \leqslant k < n$.

Remark 1

Assuming $(*)$ yields in particular $C_0(X) \cap \{0\}^k \times \mathbb{R}^{n-k} = \{0\}^n$ which is equivalent to the inclusion

$$X \cap U \subset \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid ||y|| \leqslant C||x||\},$$

for some small neighborhood U of $0 \in \mathbb{R}^n$ and some constant $C > 0$.

Remark 2

In the situation introduced above, for any $v \in \{0\}^k \times \mathbb{R}^{n-k} \cap \mathbb{S}^{n-1}$, we have $v \in V_0(X)$. Moreover, $\mathbb{R}v \cap C_0(X) = \{0\}$ which implies that considering the decomposition $\mathbb{R}^n = (\mathbb{R}v)^\perp \oplus \mathbb{R}v$, there exists $c > 0$ such that:

$$(X, 0) \subset \{z \oplus tv \in (\mathbb{R}v)^\perp \oplus \mathbb{R}v \mid |t| \leqslant c||z||\},$$

which is to be understood in the sense that a representative of the germ is contained in the set on the right-hand side.

This remark allows us to state the following definition in which we use

$$P_v(\theta, c) := \{z \oplus tv \in (\mathbb{R}v)^\perp \oplus \mathbb{R}v \mid |t| \leqslant c||z||^\theta\}$$

defined for $\theta > 0$ and $c > 0$.
We also write $V_0^C(X) := \mathbb{R}^{n-k} \cap \mathbb{S}^{n-1} \subset V_0(X)$ and call the directions $v \in V_0^C(X)$ used here *camber directions*.

Definition

Under the assumption $(*)$ we define the *directional curvature* in the camber direction $v \in V_0^C(X)$ of $(X,0)$ to be $\theta_v(X,0) := \sup\{\theta > 0 \mid \exists c > 0\colon (X,0) \subset P_v(\theta,c)\} \in [1,+\infty]$.

Lemma 1

Given $v \in \{0\}^k \times \mathbb{R}^{n-k} \cap \mathbb{S}^{n-1}$, the least upper bound $\theta_v(X,0)$ is finite, unless $(X,0) \subset (\mathbb{R}v)^\perp$.

Proof: By the Curve Selection Lemma we find a definable curve $\gamma\colon [0,\varepsilon) \to X$ with $\gamma \cap (\mathbb{R}v)^\perp = \{0\}$. Now, $\gamma(t) = (z(t) \oplus \tau(t)v) \in (\mathbb{R}v)^\perp \oplus \mathbb{R}v$ means that after identifying the direct sum with the product $H \times \mathbb{R}$ where $H = (\mathbb{R}v)^\perp$ we may reparametrize $z(t)$ by the distance to the origin. Next, having $||z(t)|| = t$ allows us to identify the curve $\gamma(t)$ with $\eta(t) = (t, \tilde{\tau}(t))$. Then it is easy to see that $\mathrm{ord}_0\eta(t)$ (minimal order of the components $\eta = (\eta_1, \ldots, \eta_n)$) is an upper bound for $\theta_v(X,0)$ and by construction this $\mathrm{ord}_0\eta(t)$ is finite.

Remark 3

Unless $(X, 0) \subset (\mathbb{R}v)^{\perp}$, the least upper bound is attained and belongs to the field of exponents $\mathbb{F}$ of the o-minimal structure (which is $\mathbb{Q}$ in the globally subanalytic or semi-algebraic cases). In fact, if we consider the two projections $\pi(z \oplus tv) = z$ and $p(z \oplus tv) = t$, we see that on $\pi^{-1}(0) \cap X \subset p^{-1}(0) \cap X$. This readily implies that these two functions satisfy on X the Łojasiewicz inquality and in fact, the number $\theta_v(X, 0)$ corresponds to their Łojasiewicz exponent, which we know to be attained.

Lemma 2

The directional curvature $\theta_v(X, 0)$ in the camber direction v is equal to

$$\sup\{\theta > 0 \mid (X, 0) \subset \{|\langle v, x\rangle| \leqslant \mathrm{const.}\|\pi(x)\|^{\theta}\}\},$$

where π denotes the orthogonal projection on $\mathrm{Aff}(C_0(X))$.

Definition

The *general curvature* of the germ $(X,0) \subset \mathbb{R}^k_x \times \mathbb{R}^{n-k}_y$ satisfying $(*)$ is defined to be

$$\theta(X,0) := \sup\left\{\theta > 0 \mid (X,0) \subset \left\{||y|| \leqslant \mathrm{const.}||x||^\theta\right\}\right\}.$$

Lemma 3

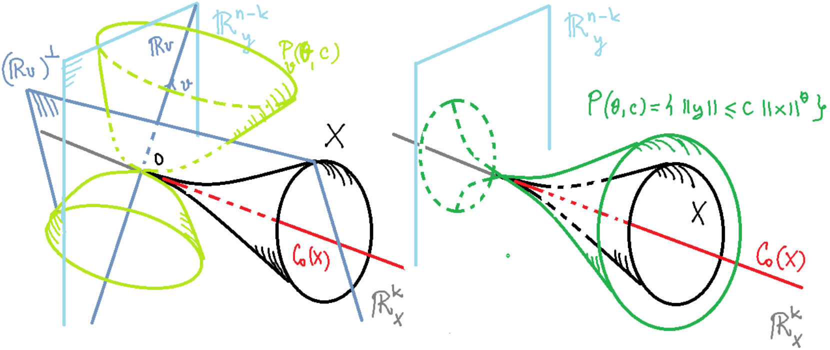
Unless $(X,0) \subset \mathbb{R}^k \times \{0\}^{n-k}$, the supremum in the definition of $\theta(X,0)$ is finite and attained in $\mathbb{F}_{\geqslant 1}$.

Proof: Put $\pi_k(x,y) = x$ and $\pi_{n-k}(x,y) = y$. Then by Remark 1, $\pi_k^{-1}(0) \cap X \subset \pi_{n-k}^{-1}(0) \cap X$ and as earlier it is easy to see that $\theta(X,0)$ corresponds to their Łojasiewicz exponent. Hence the result.

Definition

We call the germ $(X,0)$ *superquadratic* if $\theta(X,0) < 2$.

This encompasses the cases considered previously.



Thm $\sup\{\theta>0 \mid (X,0)\subset P(\theta,c),\text{ for some } c>0\} =$

$= \inf\{\sup\{\theta>0 \mid (X,0)\subset P_v(\theta,c),\text{ for some } c>0\} : v\in \{0\}^k\times\mathbb{R}^{n-k}\cap\mathbb{S}^{n-1}\}$

whenever $\operatorname{Aff}(C_0(X)) = \mathbb{R}^k\times\{0\}^{n-k}$.

Proposition

If $0 \in \mathrm{Reg}_2 X$, then $\theta(X,0) \geqslant 2$.

Proof: Since $C_0(X)$ coincides with the tangent space $T_0 X$, we conclude that $(*)$ reads $T_0 X = \mathbb{R}^k \times \{0\}^{n-k}$ with $k = \dim_0 X$. Then $(X,0)$ is the graph of a $\mathcal{C}^2$-smooth map germ $f\colon (\mathbb{R}^k,0) \to (\mathbb{R}^{n-k},0)$ of the first k variables with differential $d_0 f = 0$. Therefore, $f(x) = \frac{1}{2} d_0^2 f(x,x) + o(||x||^2)$ which implies $||f(x)|| \leqslant \mathrm{const.}||x||^2$ in a neighbourhood of the origin and the assertion follows.

Theorem 1

$\theta(X,0) = \min\{\theta_v(X,0) \mid v \in V_0^C(X)\}$.

Proof: Let $v_1, \ldots, v_{n-k} \in \mathbb{S}^{n-1}$ span $\mathrm{Aff}(C_0(X))^\perp$. We immediately obtain

$$\inf\{\theta_v(X,0) \mid v \in \mathrm{Aff}(C_0(X))^\perp \cap \mathbb{S}^{n-1}\} \leqslant \min_{i=1,\ldots,n-k}\{\theta_{v_i}(X,0)\}.$$

Conversely, take any $v \in \mathrm{Aff}(C_0(X))^\perp \cap \mathbb{S}^{n-1}$ and assume that $\min_{i=1,\dots,n-k}\{\theta_{v_i}(X,0)\} > \theta_v(X,0)$. Then there exists $\delta > 0$ such that $\theta_{v_i}(X,0) > \delta > \theta_v(X,0)$ for any $i = 1,\dots,n-k$.
Let us write $v = \sum \lambda_i v_i$ and take a decomposition of $\mathrm{Aff}(C_0(X))^\perp = U \oplus \mathbb{R}v$, denote $x \in X$ as $x = x_1 \oplus x_2 \oplus tv \in \mathrm{Aff}(C_0(X)) \oplus U \oplus \mathbb{R}v$. By Lemma 2 we can find constants c_i for which $|t| \leqslant c_i \|x_1\|^{\theta_{v_i}(X,0)}$.
We are now ready to bound the norm of $z = x_1 \oplus x_2$ as follows

$$\begin{aligned}\|z\|^\delta = \mathrm{const.} \sum \lambda_i c_i \|z\|^\delta &\geqslant \mathrm{const.} \sum \lambda_i c_i \|x_1\|^\delta \geqslant \\ &\geqslant \mathrm{const.} \sum \lambda_i \|tv_i\| \geqslant \\ &\geqslant \mathrm{const.} \|\sum \lambda_i t v_i\| = \mathrm{const.} \|tv\|.\end{aligned}$$

We obtain thus a contradiction with the choice of $\theta_v(X,0)$.

Definition

We define the *shape function* of the germ $(X,0)$ in the camber direction $v \in V_0^C(X)$ by

$$g_{X,v}(r) = \max\{|t| \mid (z \oplus tv) \in X, ||z|| = r\}$$

where we decompose as earlier $\mathbb{R}^n = (\mathbb{R}v)^\perp \oplus \mathbb{R}v$.

Of course, $g_{X,v}$ is a definable function and $(*)$ implies $g_{X,v}(0) = 0$ (cf. Remark 2). Recall that $\mathrm{ord}_0 g_{X,v}$, whenever finite, is the exponent η appearing in the expansion $g_{X,v}(r) = cr^\eta + o(r^\eta)$, where $c > 0$ is a constant. It can be characterised geometrically by $\mathrm{ord}_0 g_{X,v} = \sup\{\eta > 0 \mid g_{X,v}(r) \leqslant \mathrm{const.}r^\eta,\ \mathrm{near}\ 0\}$ the least upper bound being attained.

Lemma 4

$\mathrm{ord}_0 g_{X,v} = \theta_v(X,0)$.

Proof: Obviously, there is nothing to prove when $(X,0) \subset (\mathbb{R}v)^{\perp}$ as both sides are $+\infty$ (cf. $g_{X,v} \equiv 0$). Let us consider the remaining case. The set

$$E := \{(r, z \oplus tv) \in [0,\varepsilon) \times X \mid ||z|| = r, g_{X,v}(||z||) = |t|\}$$

is definable and has an accumulation point at the origin. Therefore, there is a continuous definable selection $\gamma \colon [0,\varepsilon) \ni r \mapsto (r, z(r) \oplus \tau(r)v) \in E$. Then $z(r)$ is automatically parameterised by arc-length: $\|z(r)\| = r$, and $\tau(r)$ being definable, we may assume it has constant sign, say $\tau(r) \geqslant 0$. It follows that $g_{X,v}(r) = \tau(r)$ so that $\mathrm{ord}_0 g_{X,v} = \mathrm{ord}_0 \tau$. Clearly, $\tau(r) \leqslant \mathrm{const.}||z(r)||^{\theta_v(X,0)}$ so that $\mathrm{ord}_0\tau \geqslant \theta_v(X,0)$. On the other hand, given $z \oplus tv \in X$ near the origin, we have

$$|t| \leqslant g_{X,v}(||z||) = \tau(||z||) \leqslant \mathrm{const.}||z||^{\mathrm{ord}_0\tau}$$

whence $\theta_v(X,0) \geqslant \mathrm{ord}_0\tau$ and we are done.

Lemma 5

For $g_X(r) := \max\{||y|| \mid (x,y) \in X, ||x|| = r\}$, $0 \leq r \ll 1$, which is a definable function, there is $\mathrm{ord}_0 g_X = \theta(X,0)$.

We are now ready for the general criterion.

Theorem 2 (Białożyt-Bysiewicz-Denkowski 2025)

If $\theta_v(X,0) < 2$, then $0 \in \overline{M_X}$ — actually,

$$0 \in \overline{M_X \cap \{(z \oplus tv) \in (\mathbb{R}v)^\perp \oplus \mathbb{R}v \mid |t| > \mathrm{const.}||z||^{\theta_v(X,0)}\}},$$

and $\dim C_0(M_X) \cap \mathbb{R}v = 1$, i.e. either v or $-v$ is tangent to M_X at the origin.

Remark

In fact, we prove $r_v(0) = 0 \ \Rightarrow \ v \in C_0(M_X)$.

Proof: Consider the selection $\gamma(r) = (r, z(r) \oplus \tau(r)v)$ from the proof of Lemma 4, with $\tau(r) \geqslant 0$. Thanks to the definability of $\xi(r) := z(r) \oplus \tau(r)v \in X$, which is non-constant, we know that the tangent cone to its image is $C_0(\xi) = \mathbb{R}_+ w$, for some $w \in C_0(X) \subset (\mathbb{R}v)^\perp$. Thus, $\langle v, w \rangle = 0$. Incidentally, $\mathbb{R}_+ w$ is the tangent half-line to the image of $z(r)$.
If we prove that the directional reaching radius $r_v(0) = 0$, then we get for the weak reaching radius, $r'(0) = 0$. Hence, the reaching radius $r(0) = 0$ and, eventually, $0 \in \overline{M_X}$ by the Reaching Radius Theorem.
Let us consider $\widehat{\tau} \colon \mathbb{R}_+ w \ni sw \to \tau(s) \in \mathbb{R}_t$, then $\mathrm{ord}_0 \widehat{\tau} = \mathrm{ord}_0 \tau$ and the latter coincides with $\theta_v(X, 0)$ as was shown in the proof of Lemma 4. Therefore, $\widehat{\tau}$ is superquadratic, which means that its graph $\Gamma_{\widehat{\tau}}$ enters any disc $D((0, \rho), \rho)$, $0 < \rho \ll 1$, cf. the Suoerquadraticity Lemma. This graph is the projection onto the plane $\mathbb{R}w \oplus \mathbb{R}v$ of the graph Γ_ξ of ξ. Eventually, by construction, we conclude that the graph of ξ enters any ball $\mathbb{B}(\rho v, \rho)$, $0 < \rho \ll 1$, so that $0 \notin m(\rho v)$, whence $r_v(0) = 0$, as required.

A careful study of $r_v(0) = 0$ shows that $v \in C_0(M_X)$: for any $t > 0$, $0 \notin m(tv)$ but there is $y(t) \in m(tv)$ and $y(t) \to 0$, when $t \to 0^+$. Obviously, $r(y(t)) \in (0, +\infty)$ and it can be shown using the proximal inequality that

$$v(t) := \frac{tv - y(t)}{||tv - y(t)||} \to v, \quad t \to 0^+.$$

Then we use the sequence $M_X \ni y(t) + r(y(t))v(t) \to 0$ to obtain $v \in C_0(M_X)$.
This ends the proof.

Corollary

If $\theta(X, 0) < 2$, then $0 \in \overline{M_X}$.

It follows directly from Theorems 1 and 2, since there must be a direction v in which $\theta_v(X, 0) < 2$.

Example

Consider the horn $X\colon x^3 = y^2 + z^2$. Its medial axis consists of the positive part of the x-axis together with a punctured revolution surface in the half-space $\{x \leqslant 0\}$. Since $C_0(X) = [0, +\infty) \times \{0\}^2$, the camber directions are $\{0\} \times \mathbb{R}^2 \cap \mathbb{S}^2$ and it is easy to see that $\theta(X, 0) = \frac{3}{2}$. The Theorem applies and indeed $0 \in \overline{M_X}$. Note however, that we completely miss the x-axis part of the medial axis which also reaches the origin. What Theorem 2 sees is the much less obvious part of the medial axis. Observe also, that in the case of a horn we cannot apply the Tangent Cone Criterion, nor any previously developed idea of superquadracity as the origin is a singular point.

Thank you for your attention!

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